ENVELOPING OPERADS
AND
BICOLOURED NONCROSSING CONFIGURATIONS

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Abstract. An operad structure on certain bicoloured noncrossing configurations in regular polygons is studied. Motivated by this study, a general functorial construction of enveloping operad, with input a coloured operad and output an operad, is presented. The operad of noncrossing configurations is shown to be the enveloping operad of a coloured operad of bubbles. Several suboperads are also investigated, and described by generators and relations.

Contents

Introduction

This article is concerned with some operads in a context of algebraic combinatorics. The theory of operads started as a device to organize the complicated structures appearing in algebraic topology, and in particular the many operations arising on loop spaces and their homology [BV73]. Since then, it has been more and more clear that operads can be used with profit in various other settings, see [Kap98, LV12] and the references therein. One striking example is the study of the moduli spaces of complex curves, where the compactification naturally involves gluing operations [GK95].

In algebraic combinatorics, there is on the one hand a long tradition of using associative algebras, words and languages to describe combinatorial objects and to decompose them into more elementary pieces. On the other hand, the theory of operads is closely related to various kinds of trees, and provides a way to create new objects by gluing smaller ones [Cha08]. One

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can therefore hope that algebraic combinatorics can benefit from a larger use of the theory of operads, and maybe there can be also fruitful interaction in the other way.

In this article, these ideas will get illustrated by some examples of operads with a combinatorial flavour, and by a general operadic construction inspired by these examples.

Let us describe our motivation for this work. In a previous article [Cha07], the first named author has considered an operad structure on the objects called noncrossing trees and noncrossing plants. These objects can be depicted as simple graphs inside regular polygons, and are some kind of noncrossing configurations that are well-known combinatorial objects [FN99, FS09]. The composition of these operads has a very simple graphical description and it is tempting and easy to generalize this composition as much as possible, by removing some constraints on the objects. This leads to a very big operad of noncrossing configurations. This research initially started as a study of this operad, with possible aim the description of its suboperads.

This study has led us to the following results. First, we introduce a general functorial construction from coloured operads to operads, which is called the *enveloping operad*. This can be compared to the amalgamated product of groups, in the sense that it takes a compound object to build a unified object in the simplest possible way, by imposing as few relations as possible. The main interest of this construction relies on the fact that a lot of properties of an enveloping operad (as *e.g.*, its Hilbert series and a presentation by generators and relations) can be obtained from its underlying coloured operad.

Next, we consider the operad BNC of bicoloured noncrossing configurations, defined by a simple graphical composition, and show that it admits a description as the enveloping operad of a very simple coloured operad on two colours called Bubble. We also obtain a presentation by generators and relations of the coloured operad Bubble.

Then this understanding of the operad BNC is used to describe in details some of its suboperads, namely those generated by two chosen generators among the binary generators of BNC. This already gives an interesting family of operads, where one can recognize some known ones: the operad of noncrossing trees [Cha07], the operad of noncrossing plants [Cha07], the dipterous operad [LR03, Zin12], and the 2-associative operad [LR06, Zin12]. Our main results here are a presentation by generators and relations for all these suboperads but one, and also the description of all the generating series. It should be noted that the presentations are obtained in a case-by-case fashion, using similar rewriting techniques.

This article is organized as follows. In Section 1, the general construction of enveloping operads is given and its properties described. Next, in Section 2, we introduce the operad BNC and prove that this operad is isomorphic to an enveloping operad. Finally, in Section 3, several suboperads of BNC are considered, in a more or less detailed way.

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1. Enveloping operads of coloured operads

The aim of this Section is threefold. We begin by recalling some basic notions about coloured operads and free coloured operads. Then, we introduce the main object of this paper: the construction which associates an operad with a coloured one, namely its enveloping operad. We finally justify the benefits of seeing an operad $\mathcal{P}$ as an enveloping operad of a coloured one $\mathcal{C}$ by reviewing some properties of $\mathcal{P}$ that can be deduced from the ones of $\mathcal{C}$.

1.1. **Free coloured operads.** Let k be a positive integer. We shall consider in this work non-symmetric k-coloured operads in the category of sets; notice that through a slight translation, all next notions and results remain valid in the category of vector spaces. Coloured operads are operads where a composition $x \circ_i y$ is defined between two elements x and y of $\mathcal{C}$ if and only if the *output colour* $\mathbf{out}(y)$ of y is the same as the ith *input colour* $\mathbf{in}_i(x)$ of x; the set of allowed colours being $[k] := \{1, \dots, k\}$. In the sequel, we only consider k-coloured operads $\mathcal{C}$ such that $\mathcal{C}(1) = \{\mathbf{1}_c : c \in [k]\}$ where $\mathbf{1}_c$ is a unit with c as input and output colour, and such that there are finitely many elements of arity n for any $n \geqslant 1$. Since (noncoloured) operads are 1-coloured operads, the following notions and notations also work for operads.

1.1.1. *Coloured syntax trees.* A *k-coloured collection* is a graded set $C = \uplus_{n \geqslant 2} C(n)$ together with two maps $\mathbf{out}, \mathbf{in}_i : C \to [k]$ which respectively associate with an element x of $C(n)$ the colour of its output and the colour of its ith input, where i lies between 1 and the arity $|x| := n$ of x.

For any k-coloured collection C, we denote by $\mathbf{Free}(C)$ the set of *k-coloured syntax trees on* C, that are planar rooted trees such that

(1) internal nodes of arity ℓ are labeled by elements of $C(\ell)$;
(2) for any internal nodes r and s such that s is the ith child of r, we have $\mathbf{in}_i(x) = \mathbf{out}(y)$ where x (resp. y) is the label of r (resp. s).

Let T be a coloured syntax tree on C. The *arity* $|T|$ of T is its number of leaves and the *degree* $\deg(T)$ of T is its number of internal nodes. The leaves of T are numbered from left to right. By a slight abuse of notation, for any internal node r of T, $\mathbf{out}(r)$ (resp. $\mathbf{in}_i(r)$) denotes the colour $\mathbf{out}(x)$ (resp. $\mathbf{in}_i(x)$) where x is the label of r. Continuing the same abuse, $\mathbf{out}(T)$ is the output colour of the root of T and $\mathbf{in}_i(T)$ is the colour of the input of the internal node on which the ith leaf of T is attached. A *corolla* labeled by $x \in C(\ell)$ is the coloured syntax tree $\mathrm{c}(x)$ on C consisting in one internal node labeled by x with ℓ leaves as children. Figure 1 shows an example of a 2-coloured syntax tree.

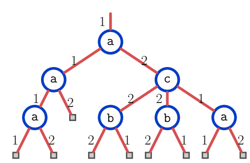
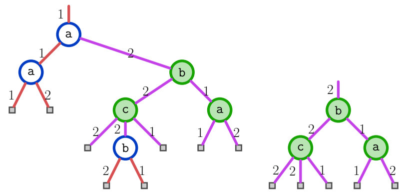


FIGURE 1. A 2-coloured syntax tree on the 2-coloured collection $C := C(2) \uplus C(3)$ defined by $C(2) := \{\mathtt{a}, \mathtt{b}\}$, $C(3) := \{\mathtt{c}\}$, $\mathbf{out}(\mathtt{a}) := 1$, $\mathbf{out}(\mathtt{b}) := 2$, $\mathbf{out}(\mathtt{c}) := 2$, $\mathbf{in}_1(\mathtt{a}) := 1$, $\mathbf{in}_2(\mathtt{a}) := 2$, $\mathbf{in}_1(\mathtt{b}) := 2$, $\mathbf{in}_2(\mathtt{b}) := 1$, $\mathbf{in}_1(\mathtt{c}) := 2$, $\mathbf{in}_2(\mathtt{c}) := 2$, $\mathbf{in}_3(\mathtt{c}) := 1$. Its arity is 9, its degree is 7, its output colour is 1 and its second input colour is 2.

We say that a coloured syntax tree S is a *subtree* of T if it is possible to fit S at a certain place of T, by possibly superimposing leaves of S and internal nodes of T. Figure 2 shows a coloured syntax tree an one of its subtrees.

In what follows, specifically to deal with presentations of coloured operads, with shall make use of rewrite rules on coloured syntax trees. A *rewrite rule* is a binary relation $\mapsto$ on coloured syntax trees where $S \mapsto T$ only if the trees S and T have the same arity. Let S' and T' be two coloured syntax trees. We say that S' can be *rewritten* by $\mapsto$ into T' if there exist two coloured syntax trees S and T satisfying $S \mapsto T$ and S' has a subtree S such that, by replacing S by T in S', we obtain T'. By a slight abuse of notation, we denote by $S' \mapsto T'$ this property. We shall



(a) T (b) S

FIGURE 2. A coloured syntax tree T on the coloured collection defined in Figure 1 and S, one of its subtrees.

use the standard terminology (*confluent*, *terminating*, *normal form*, *etc.*) about rewrite rules (see for instance [BN98]).

1.1.2. *Free coloured operads.* The set $\mathbf{Free}(C)$, together with the units $\{\mathbf{1}_c : c \in [k]\}$, is endowed with a k-coloured operad structure defined as follows. For any coloured syntax trees S and T on C, the composition $S \circ_i T$, defined when the output colour of T is the same as the ith input of S, is the coloured syntax tree obtained by grafting the root of T on the ith leaf of S. Figure 3 shows an example of such a composition. This forms the *free coloured operad generated by* C, denoted by $\mathbf{Free}(C)$.

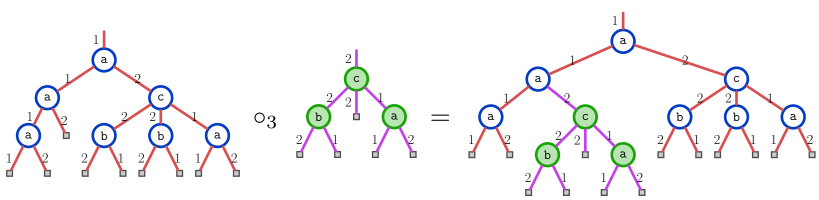


FIGURE 3. An example of a composition in the free coloured operad generated by the coloured collection defined in Figure 1.

1.2. **The construction.** Let us now introduce the construction associating a (noncoloured) operad with a coloured one. We begin by giving the formal definition of what enveloping operads of coloured operads are, and then, give a combinatorial interpretation of the construction in terms of anticoloured syntax trees.

1.2.1. *Enveloping operads.* Let $\mathcal{C}$ be a k-coloured operad. We denote by $\mathcal{C}^+$ the k-coloured collection $\mathcal{C} \setminus \mathcal{C}(1)$ and by $\bar{\mathcal{C}}$ the 1-coloured collection consisting in the elements of $\mathcal{C}^+$ with 1 as output and input colours. The *enveloping operad* $\mathbf{Hull}(\mathcal{C})$ of $\mathcal{C}$ is the smallest (noncoloured) operad containing $\mathcal{C}^+$. In other terms,

$$\mathbf{Hull}(\mathcal{C}) := \mathbf{Free}\left(\bar{\mathcal{C}}\right) /_{\equiv}, \tag{1.2.1}$$

where $\equiv$ is the smallest operadic congruence of $\mathbf{Free}\left(\bar{\mathcal{C}}\right)$ satisfying

$$\mathrm{c}(x) \circ_i \mathrm{c}(y) \equiv \mathrm{c}(x \circ_i y), \tag{1.2.2}$$

for all $x, y \in \bar{\mathcal{C}}$ such that $x \circ_i y$ is well-defined in $\mathcal{C}$.

1.2.2. *Reductions.* Let T be a 1-coloured syntax tree of $\mathbf{Free}\left(\bar{\mathcal{C}}\right)$ and e be an edge of T connecting two internal nodes r and s respectively labeled by x and y, such that s is the ith child of r and, as elements of $\mathcal{C}$, $\mathbf{in}_i(x) = \mathbf{out}(y)$. Then, the *reduction* of T with respect to e is the tree obtained by replacing r and s by an internal node labeled by $x \circ_i y$ (see Figure 4). This is another element of $\mathbf{Free}\left(\bar{\mathcal{C}}\right)$.

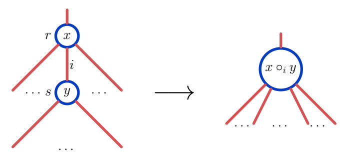


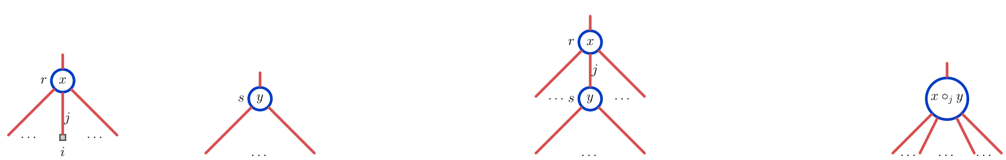
FIGURE 4. The reduction of 1-coloured syntax trees.

1.2.3. *Anticoloured syntax trees.* For any k-coloured collection C, we denote by $\mathbf{Anti}(C)$ the set of *k-anticoloured syntax trees on C*, that are planar rooted trees such that

(1) internal nodes of arity ℓ are labeled by elements of $C(\ell)$;
(2) for any internal nodes r and s such that s is the ith child of r, we have $\mathbf{in}_i(x) \neq \mathbf{out}(y)$ where x (resp. y) is the label of r (resp. s).

The same terminology as the one introduced in Section 1.1.1 for coloured syntax trees remains valid for anticoloured syntax trees.

1.2.4. *The operad of anticoloured syntax trees.* If $\mathcal{C}$ is a coloured operad, the set $\mathbf{Anti}(\mathcal{C}^+)$, together with the unit $\mathbf{1}$, is endowed with an operad structure for the composition defined as follows. Let S and T be two anticoloured syntax trees on $\mathcal{C}^+$. If $\mathbf{out}(T) \neq \mathbf{in}_i(S)$, $S \circ_i T$ is the anticoloured syntax tree obtained by grafting the root of T on the ith leaf of S. Otherwise, when $\mathbf{out}(T) = \mathbf{in}_i(S)$, $S \circ_i T$ is the anticoloured syntax tree obtained by grafting the root of T on the ith leaf of S and then, by reducing the obtained tree with respect to the edge connecting the nodes r and s, where r is the parent of the ith leaf of S and s is the root of T. (see Figure 5).



(a) S (b) T (c) $S \circ_i T$ if $\mathbf{out}(T) \neq \mathbf{in}_i(S)$ (d) $S \circ_i T$ if $\mathbf{out}(T) = \mathbf{in}_i(S)$

FIGURE 5. The two cases for the composition of two anticoloured trees S and T.

Proposition 1.1. *For any coloured operad $\mathcal{C}$, the operads $\mathbf{Hull}(\mathcal{C})$ and $\mathbf{Anti}(\mathcal{C}^+)$ are isomorphic.*

Proof. Let $\phi : \mathbf{Hull}(\mathcal{C}) \to \mathbf{Anti}(\mathcal{C}^+)$ be the map associating with any $\equiv$-equivalence class of 1-coloured syntax trees on $\bar{\mathcal{C}}$, the only anticoloured syntax tree on $\mathcal{C}^+$ belonging to it. To prove the statement, let us show that ϕ is a well-defined operad isomorphism.

For that, consider the rewrite rule $\mapsto$ on the 1-coloured syntax trees on $\bar{\mathcal{C}}$ by setting $S \mapsto T$ if T can be obtained from S by a reduction. Operadic axioms ensure that $\mapsto$ is confluent, and since any rewriting decreases the number of internal nodes, $\mapsto$ is terminating. The normal forms of $\mapsto$ are the trees that cannot be reduced, and thus, are anticoloured syntax trees on $\mathcal{C}^+$. Since by definition of $\equiv$, $S \mapsto T$ implies $S \equiv T$, the application ϕ is well-defined and is a bijection.

Finally, let $[S]_\equiv, [T]_\equiv \in \mathbf{Hull}(\mathcal{C})$, $S := \phi([S]_\equiv)$, and $T := \phi([T]_\equiv)$. The only anticoloured syntax tree in $[S \circ_i T]_\equiv$ is obtained by grafting S and T together and performing, if possible, a reduction with respect to the edge linking these. Since the obtained tree is also the anticoloured syntax tree $S \circ_i T$ of $\mathbf{Anti}(\mathcal{C}^+)$, ϕ is an operad morphism. $\square$

Proposition 1.1 implies that the elements of $\mathbf{Hull}(\mathcal{C})$ can be regarded as anticoloured trees, endowed with their composition defined above. We shall maintain this point of view in the rest of this paper by setting $\mathbf{Hull}(\mathcal{C}) := \mathbf{Anti}(\mathcal{C}^+)$.

1.2.5. *Functoriality.* Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two k-coloured operads. Recall that a map $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ is a *coloured operad morphism* if it preserves the arities, commutes with compositions maps and, for any $x, y \in \mathcal{C}_1$ and $i \in [|x|]$, if the composition $x \circ_i y$ is defined in $\mathcal{C}_1$, then the composition $\phi(x) \circ_i \phi(y)$ is defined in $\mathcal{C}_2$.

Given $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ a coloured operad morphism, let the map $\mathbf{Hull}(\phi) : \mathbf{Hull}(\mathcal{C}_1) \to \mathbf{Hull}(\mathcal{C}_2)$ be the unique operad morphism satisfying

$$\mathbf{Hull}(\phi)(\mathrm{c}(x)) := \mathrm{c}(\phi(x)) \tag{1.2.3}$$

for any $x \in \mathcal{C}_1$.

Theorem 1.2. *The construction* **Hull** *is a functor from the category of coloured operads to the category of operads that preserves injections and surjections.*

Proof. For any coloured operad $\mathcal{C}$, $\mathbf{Hull}(\mathcal{C})$ is by definition an operad on anticoloured syntax trees on $\mathcal{C}^+$. Moreover, by induction on the number of internal nodes of the anticoloured syntax trees, it follows that for any coloured operad morphism ϕ, $\mathbf{Hull}(\phi)$ is a well-defined operad morphism.

Since **Hull** is compatible with map composition and sends the identity coloured operad morphism to the identity operad morphism, **Hull** is a functor. It is moreover plain that if ϕ is an injective (resp. surjective) coloured operad morphism, then $\mathbf{Hull}(\phi)$ is an injective (resp. surjective) operad morphism. $\square$

Theorem 1.2 is rich in consequences: Propositions 1.4, 1.5, 1.7, 1.6 of next Section directly rely on it.

Notice that **Hull** is a surjective functor. Indeed, since an anticoloured syntax tree on a 1-coloured collection is necessarily a corolla, for any operad $\mathcal{P}$, $\mathbf{Hull}(\mathcal{P})$ contains only corollas labeled on $\mathcal{P}^+$ and it is therefore isomorphic to $\mathcal{P}$.

Notice also that **Hull** is not an injective functor. Let us exhibit two 2-coloured operads not themselves isomorphic that produce by **Hull** two isomorphic operads. Let $\mathcal{C}_1$ be the 2-coloured operad where $\mathcal{C}_1(2) := \{\alpha_2\}$ with $\mathbf{out}(\alpha_2) := 1$ and $\mathbf{in}_1(\alpha_2) := \mathbf{in}_2(\alpha_2) := 2$. and for all $n \geqslant 3$, $\mathcal{C}_1(n) := \emptyset$. Due to the output and input colours of α_2, there are not nontrivial compositions in $\mathcal{C}_1$. On the other hand, let $\mathcal{C}_2$ be the 2-coloured operad where, for all $n \geqslant 2$, $\mathcal{C}_2(n) := \{\beta_n\}$ with $\mathbf{out}(\beta_n) := 1$, $\mathbf{in}_1(\beta_n) := 1$, and $\mathbf{in}_i(\beta_n) := 2$ for all $2 \leqslant i \leqslant n$. Nontrivial compositions of $\mathcal{C}_2$ are only defined for the first position by $\beta_n \circ_1 \beta_m := \beta_{n+m-1}$, for any $n, m \geqslant 2$. One observes that $\mathbf{Hull}(\mathcal{C}_1)$ and $\mathbf{Hull}(\mathcal{C}_2)$ are both the free operad generated by one element of arity 2 with no nontrivial relations, and hence, are isomorphic. The isomorphism between $\mathbf{Hull}(\mathcal{C}_1)$ and $\mathbf{Hull}(\mathcal{C}_2)$ can be described by a left-child right-sibling bijection [CLRS09] between binary trees and planar rooted trees.

1.3. **Bubble decompositions of operads and consequences.** Let $\mathcal{P}$ be an operad. We say that $\mathcal{C}$ is a *k-bubble decomposition* of $\mathcal{P}$ if $\mathcal{C}$ is a k-coloured operad such that $\mathbf{Hull}(\mathcal{C})$ and $\mathcal{P}$ are isomorphic. In this case, we say that the elements of $\mathcal{C}$ are *bubbles*. As we shall show, since a bubble decomposition $\mathcal{C}$ of an operad $\mathcal{P}$ contains a lot of information about $\mathcal{P}$, the study of $\mathcal{P}$ can be confined to the study of $\mathcal{C}$.

1.3.1. *Hilbert series.* The *coloured Hilbert series* of $\mathcal{C}$ are the commutative series $\mathrm{B}_c(z_1, \dots, z_k)$, $c \in [k]$, defined by

$$\mathrm{B}_c(z_1, \dots, z_k) := \sum_{\substack{x \in \mathcal{C}^+ \\ \mathbf{out}(x) = c}} \prod_{1 \leqslant i \leqslant |x|} z_{\mathbf{in}_i(x)}. \tag{1.3.1}$$

The coefficient of $z_1^{\alpha_1} \dots z_k^{\alpha_k}$ in $\mathrm{B}_c(z_1, \dots, z_k)$ counts the nontrivial elements of $\mathcal{C}$ having c as input colour and α_d inputs of colour d for all $d \in [k]$.

As a side remark, note that one could also define as well non-commutative analogues of these Hilbert series, where one would remember the order of the input colors.

When $\mathcal{C}$ is an operad (or equivalently, a 1-coloured operad), its *Hilbert series* is

$$\mathrm{F}(t) := t + \mathrm{B}_1(t) = \sum_{n \geqslant 1} \#\mathcal{C}(n)\, t^n. \tag{1.3.2}$$

Proposition 1.3. *Let $\mathcal{C}$ be a k-coloured operad. Then, the Hilbert series $\mathrm{F}(t)$ of the enveloping operad of $\mathcal{C}$ satisfies*

$$\mathrm{F}(t) = t + \mathrm{F}_1(t) + \dots + \mathrm{F}_k(t), \tag{1.3.3}$$

where for all $c \in [k]$, the series $\mathrm{F}_c(t)$ satisfy

$$\mathrm{F}_c(t) = \mathrm{B}_c(\mathrm{F}(t) - \mathrm{F}_1(t), \dots, \mathrm{F}(t) - \mathrm{F}_k(t)).$$

Proof. Since the elements of the enveloping operad of $\mathcal{C}$ are the anticoloured syntax trees on $\mathcal{C}^+$, for all $c \in [k]$, the series $\mathrm{F}_c(t)$ are the series counting the anticoloured syntax trees on $\mathcal{C}^+$ having c as output colour. The Hilbert series of the enveloping operad of $\mathcal{C}$ is the sum of the $\mathrm{F}_c(t)$ plus t in order to count the unit. □

Note that Proposition 1.3 implies that, if the coloured Hilbert series of $\mathcal{C}$ are algebraic, the Hilbert series of $\mathbf{Hull}(\mathcal{C})$ also is. Nevertheless, as we shall see, rationality is not preserved.

1.3.2. *Suboperads and quotients.* A k-coloured operad $\mathcal{C}'$ is a *coloured suboperad* of $\mathcal{C}$ if for all $n \geqslant 1$, $\mathcal{C}'(n)$ is a subset of $\mathcal{C}(n)$ and the units of $\mathcal{C}'$ are the same as those of $\mathcal{C}$.

Proposition 1.4. *Let $\mathcal{C}$ a coloured operad and $\mathcal{C}'$ be one of its coloured suboperads (resp. quotients). Then, the enveloping operad of $\mathcal{C}'$ is a suboperad (resp. quotient) of the enveloping operad of $\mathcal{C}$.*

1.3.3. *Generating sets.* A set G of elements of $\mathcal{C}$ is a *generating set* of $\mathcal{C}$ if the smallest coloured suboperad of $\mathcal{C}$ containing G is $\mathcal{C}$ itself and if moreover G is minimal with respect to inclusion for this property. Notice that the generating set of $\mathcal{C}$ is unique, given by elements that cannot be written as a non-trivial composition. Any element x of $\mathcal{C}$ can be (non necessarily uniquely) written as $x = y \circ_i g$ where $y \in \mathcal{C}$, $i \in [|y|]$, and $g \in G$.

Proposition 1.5. *Let $\mathcal{C}$ be a coloured operad generated by G. Then, the enveloping operad of $\mathcal{C}$ is generated by*

$$\mathbf{Hull}(G) := \{\mathrm{c}(g) : g \in G\}. \tag{1.3.4}$$

1.3.4. *Symmetries.* A *symmetry* of $\mathcal{C}$ is either a coloured operad automorphism or a coloured operad antiautomorphism. A *coloured operad antiautomorphism* is a bijective map preserving the arities and, for any $x, y \in \mathcal{C}$ and $i \in [|x|]$, if the composition $x \circ_i y$ is defined, then the composition $\phi(x) \circ_{|x|-i+1} \phi(y)$ also is, and $\phi(x \circ_i y) = \phi(x) \circ_{|x|-i+1} \phi(y)$. The symmetries of $\mathcal{C}$ form a group for the composition, called the *group of symmetries* of $\mathcal{C}$.

Proposition 1.6. *Let $\mathcal{C}$ be a coloured operad and $\mathfrak{G}$ its group of symmetries. Then, the group of symmetries of the enveloping operad of $\mathcal{C}$ is $\mathbf{Hull}(\mathfrak{G})$ where*

$$\mathbf{Hull}(\mathfrak{G}) := \{\mathbf{Hull}(\phi) : \phi \in \mathfrak{G}\}. \tag{1.3.5}$$

1.3.5. *Presentations by generators and relations.* A *presentation by generators and relations* of $\mathcal{C}$ is a pair $(G, \leftrightarrow)$ where G is a k-coloured collection and $\leftrightarrow$ is the finest equivalence relation on $\mathbf{Free}(G)$ such that $\mathcal{C}$ is isomorphic to $\mathbf{Free}(G)/_{\equiv}$, $\equiv$ being the finest coloured operadic congruence containing $\leftrightarrow$.

Proposition 1.7. *Let $\mathcal{C}$ be a coloured operad admitting the presentation $(G, \leftrightarrow)$. Then, the enveloping operad of $\mathcal{C}$ admits the presentation $(\mathbf{Hull}(G), \leftrightarrow')$, where*

$$S' \leftrightarrow' T' \quad \textit{if and only if} \quad S \leftrightarrow T, \tag{1.3.6}$$

where S' (resp. T') is the coloured syntax tree on $\mathbf{Hull}(G)$ obtained by replacing any node labeled by x of S (resp. T) by $\mathrm{c}(x)$.

2. The operad of bicoloured noncrossing configurations

In this Section, we shall define an operad over a new kind of noncrossing configurations. To study it and apply the results of Section 1, we shall see it as an enveloping operad of a coloured one.

2.1. Bicoloured noncrossing configurations.

Let us start by introducing our new combinatorial object, some of its properties, and its operadic structure.

2.1.1. *Regular polygons.* Let $\mathfrak{C}$ be a regular polygon with $n+1$ vertices. The vertices of $\mathfrak{C}$ are numbered in the clockwise direction from 1 to $n+1$. An *arc* of $\mathfrak{C}$ is a tuple (i, j) with $1 \leqslant i < j \leqslant n+1$. We call *diagonal* any arc (i, j) different from $(i, i+1)$ and $(1, n+1)$, and *edge* any arc of the form $(i, i+1)$ or $(i, n+1)$. The *ith edge* of $\mathfrak{C}$ is the edge $(i, i+1)$. The edge $(1, n+1)$ is the *base* of $\mathfrak{C}$ (drawn at bottommost).

2.1.2. *Bicoloured noncrossing configurations.* A *bicoloured noncrossing configuration* (abbreviated as *BNC*) of size n is a regular polygon $\mathfrak{C}$ with $n+1$ vertices, together with two sets of arcs: a set $\mathfrak{C}_b$ of *blue arcs* (drawn as thick lines) and a set $\mathfrak{C}_r$ of *red arcs* (drawn as dotted lines). If $(i,j) \in \mathfrak{C}_b$ (resp. $(i,j) \in \mathfrak{C}_r$), we say that (i,j) is *blue* (resp. *red*). Otherwise, when $(i,j) \notin \mathfrak{C}_b \cup \mathfrak{C}_r$, we say that (i,j) is *uncoloured*. These two sets have to satisfy the three following properties:

(1) any arc is either blue, red, or uncoloured;
(2) no blue or red arc crosses another blue or red arc;
(3) all red arcs are diagonals.

We say that $\mathfrak{C}$ is *based* if its base is blue and *nonbased* otherwise. Besides, we impose by definition that there is only one BNC of size 1: the segment consisting in one blue arc. Figure 6 shows a BNC.

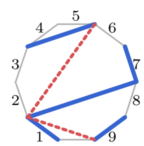


FIGURE 6. A nonbased BNC of size 9. Its blue arcs are $(1,2)$, $(2,8)$, $(4,6)$, $(7,8)$, and $(9,10)$, and its red arcs are $(2,6)$, $(2,10)$. The edges, different from the basis, are numbered.

2.1.3. *Borders.* When the size of $\mathfrak{C}$ is not smaller than 2, the *border* of $\mathfrak{C}$ is the word $\mathrm{b}(\mathfrak{C})$ of length n such that, for any $i \in [n]$, $\mathrm{b}(\mathfrak{C})_i := 1$ if the ith edge of $\mathfrak{C}$ is uncoloured and $\mathrm{b}(\mathfrak{C})_i := 2$ otherwise. For instance, the border of the BNC of Figure 6 is 211111212.

2.1.4. *Operad structure.* From now, the *arity* $|\mathfrak{C}|$ of a BNC $\mathfrak{C}$ is its size. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two BNCs of respective arities n and m, and $i \in [n]$. The composition $\mathfrak{C} \circ_i \mathfrak{D} =: \mathfrak{E}$ is obtained by gluing the base of $\mathfrak{D}$ onto the ith edge of $\mathfrak{C}$, and then,

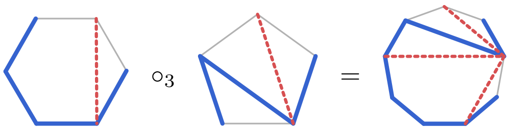
(1) if the base of $\mathfrak{D}$ and the ith edge of $\mathfrak{C}$ are both uncoloured, the arc $(i, i+m)$ of $\mathfrak{E}$ becomes red;
(2) if the base of $\mathfrak{D}$ and the ith edge of $\mathfrak{C}$ are both blue, the arc $(i, i+m)$ of $\mathfrak{E}$ becomes blue;
(3) otherwise, the base of $\mathfrak{D}$ and the ith edge of $\mathfrak{C}$ have different colours; in this case, the arc $(i, i+m)$ of $\mathfrak{E}$ is uncoloured.

For aesthetic reasons, the resulting shape is reshaped to form a regular polygon. Figure 7 shows examples of composition of BNCs.

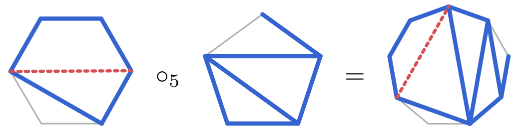
Proposition 2.1. *The set of the BNCs, together with the composition map $\circ_i$ and the BNC of arity* 1 *as unit form an operad, denoted by* BNC.

2.2. **The coloured operad of bubbles.** We now define a coloured operad involving particular BNCs and perform a complete study of it.

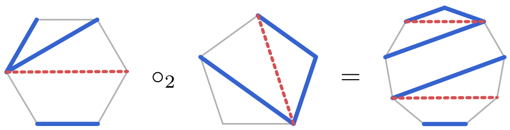
2.2.1. *Bubbles.* A *bubble* is a BNC of size no smaller than 2 with no diagonal (hence the name). Figure 8 shows an example of a bubble.



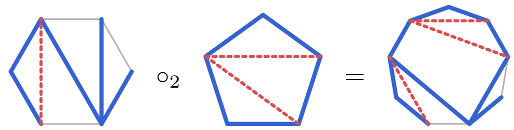
(a) Uncoloured edge over an uncoloured edge.



(b) Blue edge over a blue edge.



(c) Uncoloured edge over a blue edge.



(d) Blue edge over an uncoloured edge.

FIGURE 7. Four examples of composition in the operad BNC.

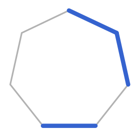


FIGURE 8. A based bubble of size 6. Its border is 111221.

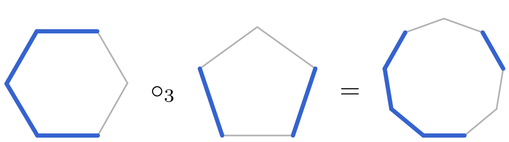
2.2.2. *Coloured operad structure.* Let $\mathfrak{B}$ be a bubble of arity n. Let us assign input an output colours to $\mathfrak{B}$ in the following way. The output colour $\mathbf{out}(\mathfrak{B})$ of $\mathfrak{B}$ is 1 if $\mathfrak{B}$ is based and 2 otherwise, and the colour $\mathbf{in}_i(\mathfrak{B})$ of the ith input of $\mathfrak{B}$ is the ith letter of the border of $\mathfrak{B}$.

Let us denote by $\mathbf{1}_1$ and $\mathbf{1}_2$ two virtual bubbles of arity 1 such that $\mathbf{out}(\mathbf{1}_1) := \mathbf{in}_1(\mathbf{1}_1) := 1$ and $\mathbf{out}(\mathbf{1}_2) := \mathbf{in}_i(\mathbf{1}_2) := 2$.

Proposition 2.2. *The set of bubbles, together with the composition map $\circ_i$ of* BNC *and the units $\mathbf{1}_1$ and $\mathbf{1}_2$ form a 2-coloured operad, denoted by* Bubble.

Proof. Since, as sets, $\mathsf{Bubble} \subseteq \mathsf{BNC}$ we only have to prove that, when defined, the composition $\mathfrak{B}_1 \circ_i \mathfrak{B}_2$ of two bubbles $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is a bubble. Since this composition is defined only if $\mathbf{out}(\mathfrak{B}_2) = \mathbf{in}_i(\mathfrak{B}_1)$, there are two possibilities: either the base of $\mathfrak{B}_2$ is blue and the ith edge of $\mathfrak{B}_1$ is uncoloured, or the base of $\mathfrak{B}_2$ is uncoloured and the ith edge of $\mathfrak{B}_2$ is blue. In both cases, no diagonal is added, and hence, $\mathfrak{B}_1 \circ_i \mathfrak{B}_2$ is a bubble. □

Notice that any bubble $\mathfrak{B}$ is wholly encoded by the pair $(\mathbf{out}(\mathfrak{B}), (\mathbf{in}_i(\mathfrak{B}))_{i \in |\mathfrak{B}|})$. Therefore, Bubble is a very simple operad: for any n, the set of elements of arity n is $[2] \times [2]^n$ and the composition, when defined, is a substitution in words. Figure 9 shows a composition in Bubble.



(a) A composition of two bubbles.

$$(1, 22211) \circ_3 (2, 2112) = (1, 22211211)$$

(b) The output and input colours of the bubbles.

FIGURE 9. A composition in the 2-coloured operad Bubble.

2.2.3. *Coloured Hilbert series.* Since Bubble contains by definition all the bubbles, the coloured Hilbert series of Bubble satisfy

$$\mathrm{B}_1(z_1, z_2) = \mathrm{B}_2(z_1, z_2) = \sum_{n \geqslant 2} (z_1 + z_2)^n = \frac{(z_1 + z_2)^2}{1 - z_1 - z_2}. \tag{2.2.1}$$

2.2.4. *Generating set.*

Proposition 2.3. *The set*

$$G_{\mathsf{Bubble}} := \{ \ldots \} \tag{2.2.2}$$

of bubbles of arity 2 *is the generating set of* Bubble*.*

Proof. Let us proceed by induction on the arity n of the bubble $\mathfrak{B}$ we want to generate. If $n = 2$, since the set of of bubbles of arity 2 is G_{Bubble}, $\mathfrak{B}$ is generated by G_{Bubble}. If $n \geqslant 3$, let $\mathfrak{B}'$ be the bubble obtained from $\mathfrak{B}$ by removing its last edge. Now, $\mathfrak{B}'$ is a bubble of arity $n - 1$, and, by induction hypothesis, $\mathfrak{B}'$ is generated by G_{Bubble}. Since, for an appropriate bubble g of G_{Bubble}, $\mathfrak{B} = \mathfrak{B}' \circ_{n-1} g$, $\mathfrak{B}$ is generated by G_{Bubble}. □

2.2.5. *Symmetries.* The *complementary* $\mathrm{cpl}(\mathfrak{B})$ of a bubble $\mathfrak{B}$ is the bubble obtained by swapping the colours of the edges of $\mathfrak{B}$. The *returned* $\mathrm{ret}(\mathfrak{B})$ of $\mathfrak{B}$ is the bubble obtained by applying on $\mathfrak{B}$ the reflection through the vertical line passing by its base. Figure 10 shows examples of these symmetries.

(a) Complementary. (b) Returned.

FIGURE 10. The complementary and the returned of a bubble.

Proposition 2.4. *The group of symmetries of* Bubble *is generated by* cpl *and* ret *and satisfies the relations*

$$\mathrm{ret} = \mathrm{ret}^{-1}, \quad \mathrm{cpl} = \mathrm{cpl}^{-1}, \quad \mathrm{ret}\,\mathrm{cpl} = \mathrm{cpl}\,\mathrm{ret}\,. \tag{2.2.3}$$

Proof. Since, by Proposition 2.3, Bubble is generated by G_{Bubble}, any symmetry of Bubble is *a fortiori* a bijection on G_{Bubble}. By computer exploration, let us consider the 8! bijections and keep only the ones that are still well-defined as coloured operad morphisms or coloured operad antimorphisms in arity three.

There are exactly two bijections that are operad morphisms up to arity three: the trivial one and the bijection α sending any $x \in G_{\mathsf{Bubble}}$ to $\mathrm{cpl}(x)$. By induction on the arity, it follows that there is a unique coloured operad morphism coinciding with α in arity two and it is cpl. Then, since cpl is a bijection, cpl is an automorphism of Bubble.

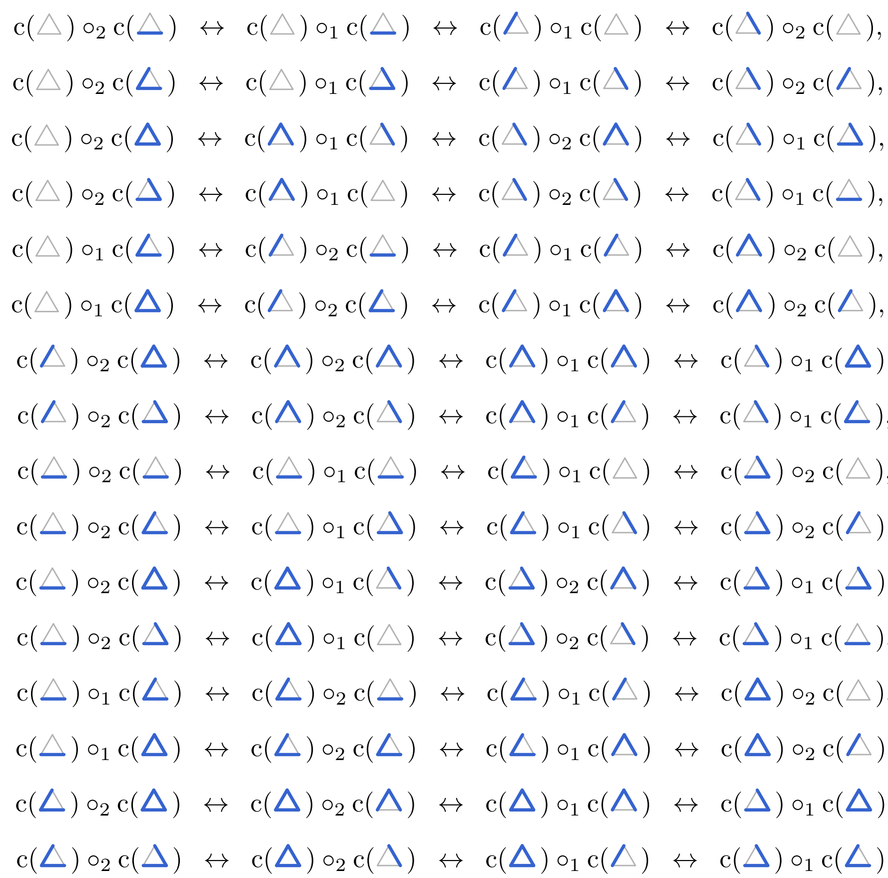
There are exactly two bijections that are operad antimorphisms up to arity three: the bijection β sending any $x \in G_{\mathsf{Bubble}}$ to $\mathrm{ret}(x)$ and the bijection γ sending any $x \in G_{\mathsf{Bubble}}$ to $\mathrm{ret}(\mathrm{cpl}(x))$. Again by induction on the arity, it follows that there is a unique coloured operad antimorphism coinciding with β (resp. γ) in arity two and it is ret (resp. $\mathrm{ret}\,\mathrm{cpl}$). Then, since ret and $\mathrm{ret}\,\mathrm{cpl}$ are bijections, ret and $\mathrm{ret}\,\mathrm{cpl}$ are antiautomorphisms of Bubble.

We have shown that the identity, cpl, ret and ret cpl are the only elements of the group of symmetries of Bubble. Relations (2.2.3) between these are obvious. □

2.2.6. *Presentation by generators and relations.*

Theorem 2.5. *The 2-coloured operad* Bubble *admits the presentation* $(G_{\mathsf{Bubble}}, \leftrightarrow)$ *where* $\leftrightarrow$ *is the equivalence relation satisfying*

(2.2.4), (2.2.5), (2.2.6), (2.2.7), (2.2.8), (2.2.9), (2.2.10), (2.2.11), (2.2.12), (2.2.13), (2.2.14), (2.2.15), (2.2.16), (2.2.17), (2.2.18), (2.2.19)

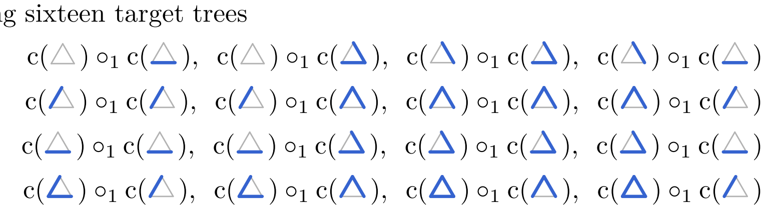


Proof. To prove the presentation of the statement, we shall show that there exists an operad isomorphism $\phi : \mathbf{Free}(G_{\mathsf{Bubble}})/_{\equiv} \to \mathsf{Bubble}$.

Let us set $\phi([\mathrm{c}(g)]_{\equiv}) := g$ for any g of G_{Bubble}. We observe that for any relation $\mathrm{c}(x) \circ_i \mathrm{c}(y) \leftrightarrow \mathrm{c}(z) \circ_j \mathrm{c}(t)$ of the statement, we have $x \circ_i y = z \circ_j t$. It then follows that ϕ can be uniquely extended into a coloured operad morphism. Moreover, since the image of ϕ contains all the generators of Bubble, ϕ is surjective.

Let us now prove that ϕ is a bijection. For that, let us orient the relation $\leftrightarrow$ by means of the rewrite rule $\mapsto$ on the coloured syntax trees on G_{Bubble} satisfying $S \mapsto T$ if $S \leftrightarrow T$ and T is one of the following sixteen target trees

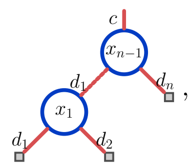
(2.2.20)



The target trees of $\mapsto$ are the only left comb trees appearing in each $\leftrightarrow$-equivalence class of the statement such that the colour of the first input of the root is the same as the colour of the first input of its child.

Let us prove that $\mapsto$ is terminating. Let ψ be the map associating the pair $(a(T), b(T))$ with a coloured syntax tree T, where $a(T)$ is the sum, for each internal node x of T, of the number of internal nodes in the tree rooted at the right child of x, and $b(t)$ is the number of internal nodes x of T having an internal node y as left child such that $\mathbf{in}_1(x) \neq \mathbf{in}_1(y)$. We observe that, for any trees T_0 and T_1 such that $T_0 \mapsto T_1$, $\psi(T_1)$ is lexicographically smaller that $\psi(T_0)$. Hence, $\mapsto$ is terminating.

The normal forms of $\mapsto$ are the coloured syntax trees on G_{Bubble} that have no subtrees S where the S are the trees appearing as a left members of $\mapsto$. These are left comb trees T such that for all internal nodes x and y of T, $\mathbf{in}_1(x) = \mathbf{in}_1(y)$. Pictorially, T is of the form

(2.2.21) $T =$ 

,

where $c \in [2]$, $d_i \in [2]$ for all $i \in [n]$, and $x_j \in G_{\mathsf{Bubble}}$ for all $j \in [n-1]$. Since T is a coloured syntax tree, given c and the d_i, there is exactly one possibility for all the x_j. Therefore, there are $f_c(n) := 2^n$ normal forms of $\mapsto$ of arity n with c as output colour. This imply that $\mathbf{Free}(G_{\mathsf{Bubble}})/_{\equiv}$ contains at most $f_c(n)$ elements of arity n and c as output colour. Then, since $f_c(n)$ is also the number of elements of Bubble with arity n and c as output colour (see Section 2.2.3), ϕ is a bijection. □

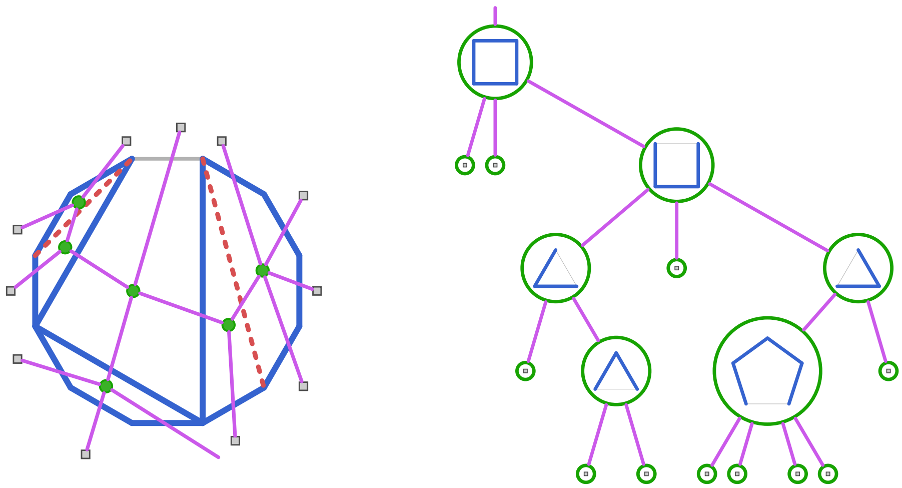
2.3. **Properties of the operad of bicoloured noncrossing configurations.** Let us come back on the study of the operad BNC. We show here that BNC is the enveloping operad of Bubble and then, by using the results of Section 2.2 together with the ones of Section 1.3, give some of its properties.

2.3.1. *Bubble decomposition.* Let $\mathfrak{C}$ be a BNC. An *area* of $\mathfrak{C}$ is a maximal component of $\mathfrak{C}$ without coloured diagonals and bounded by coloured arcs or by uncoloured edges. Any area a of $\mathfrak{C}$ defines a bubble $\mathfrak{B}$ consisting in the edges of a. The base of $\mathfrak{B}$ is the only edge of a that splits $\mathfrak{C}$ in two parts where one contains the base of $\mathfrak{C}$ and the other contains a. Blue edges of a remain blue edges in $\mathfrak{B}$ and red edges of a become uncoloured edges in $\mathfrak{B}$.

The *dual tree* of $\mathfrak{C}$ is the planar rooted tree labeled by bubbles defined as follows. If $\mathfrak{C}$ is of size 1, its dual tree is the leaf. Otherwise, put an internal node in each area of $\mathfrak{C}$ and connect any pair of nodes that are in adjacent areas. Put also leaves outside $\mathfrak{C}$, one for each edge, except the base, and connect these with the internal nodes of their adjacent areas. This forms a tree rooted at the node of the area containing the base of $\mathfrak{C}$. Finally, label each internal node of the tree by the bubble associated with the area containing it. Figure 11 shows an example of a BNC and its dual tree.

Lemma 2.6. *Let $\mathfrak{C}$ be a BNC. The dual tree of $\mathfrak{C}$ is an anticoloured syntax tree on* Bubble^+.

Proof. This follows from the definition of dual trees and the fact that a blue (resp. uncoloured) edge of a bubble $\mathfrak{B}$ is, by definition, of colour 1 (resp. 2) if it is the base of $\mathfrak{B}$ and of colour 2 (resp. 1) otherwise. □



(a) $\mathfrak{C}$

(b) The dual tree of $\mathfrak{C}$.

FIGURE 11. A bicoloured noncrossing configuration and its dual tree.

Lemma 2.7. *The map between the BNCs of arity n and the anticoloured syntax trees on* Bubble^+ *or arity n sending a BNC to its dual tree is a bijection.*

Proof. By Lemma 2.6, this map is well-defined. Let T be an anticoloured syntax tree on Bubble^+. By seeing T as a 1-coloured syntax tree, one can perform reductions in T up to obtain a corolla labeled by a BNC x. The fact that BNC is an operad ensures that the reductions can be made in any order. Thanks to the definition of the composition of BNC together with the definition of dual trees, the application sending T to x is the inverse of the map of the statement. □

Theorem 2.8. *The 2-coloured operad* Bubble *is a 2-bubble decomposition of the operad* BNC.

Proof. This is a consequence of Lemmas 2.6 and 2.7: the elements of BNC are anticoloured syntax trees on Bubble^+ and the composition of BNC translates faithfully into the composition of $\mathbf{Hull}(\mathsf{Bubble})$. □

2.3.2. *Enumeration of the bicoloured noncrossing configurations.* By using the fact that, by Theorem 2.8, Bubble is a 2-bubble decomposition of BNC, together with Proposition 1.3 and the coloured Hilbert series (2.2.1) of Bubble, we obtain the following algebraic equation for the generating series of the BNCs.

Proposition 2.9. *The Hilbert series* F *of* BNC *satisfies*

$$-t - t^2 + (1-4t)\,\mathrm{F} - 3\,\mathrm{F}^2 = 0. \tag{2.3.1}$$

First numbers of BNCs by size are

$$1,\ 8,\ 80,\ 992,\ 13760,\ 204416,\ 3180800,\ 51176960,\ 844467200. \tag{2.3.2}$$

One can refine the above enumeration of BNCs in the following way. Let us add two variables y_1 and y_2 in the system of the statement of Proposition 1.3 for BNC to obtain

$$\begin{aligned} \mathrm{F} &= t + \mathrm{F}_1 + \mathrm{F}_2 \\ \mathrm{F}_1 &= y_1 \frac{(2\,\mathrm{F} - \mathrm{F}_1 - \mathrm{F}_2)^2}{1 - 2\,\mathrm{F} + \mathrm{F}_1 + \mathrm{F}_2} \\ \mathrm{F}_2 &= y_2 \frac{(2\,\mathrm{F} - \mathrm{F}_1 - \mathrm{F}_2)^2}{1 - 2\,\mathrm{F} + \mathrm{F}_1 + \mathrm{F}_2}. \end{aligned} \tag{2.3.3}$$

By solving this system, we find that the generating series $\mathrm{F}(t, y_1, y_2)$ satisfies

$$-t + t^2 - y_1 t^2 - y_2 t^2 + (1 - 2y_1 t - 2y_2 t)\,\mathrm{F} + (-1 - y_1 - y_2)\,\mathrm{F}^2 = 0. \tag{2.3.4}$$

The parameter y_1 (resp. y_2) counts the internal nodes of anticoloured trees on Bubble^+ that are labeled by based (resp. nonbased) bubbles. By a direct translation on the BNCs themselves, y_1 counts the blue diagonals (where a blue base counts as a blue diagonal) and y_2 counts the red diagonals (where an uncoloured base counts as a red diagonal). We obtain

$$\begin{aligned} \mathrm{F}(t, y_1, y_2) &= t + 4(y_1 + y_2)t^2 + 8(y_1 + 2y_1^2 + 4y_1y_2 + 2y_2^2 + y_2)t^3 \\ &\quad + 16(y_1 + 5y_1^2 + 5y_1^3 + 15y_1^2y_2 + 10y_1y_2 + 15y_1y_2^2 + 5y_2^3 + 5y_2^2 + y_2)t^4 + \cdots . \end{aligned} \tag{2.3.5}$$

Since, by definition of the dual trees, there is a correspondence between the areas of a BNC and the internal nodes of its dual tree, the specialization $\mathrm{F}(t, y) := \mathrm{F}(t, y, y)$ satisfying

$$-t + (1 - 2y)t^2 + (1 - 4yt)\,\mathrm{F} + (-1 - 2y)\,\mathrm{F}^2 = 0 \tag{2.3.6}$$

counts the BNCs by their areas. We have

(2.3.7)
$$\begin{aligned} \mathrm{F}(t, y) = t + 8yt^2 + 16(y + 4y^2)t^3 + 32(y + 10y^2 + 20y^3)t^4 + 64(y + 18y^2 + 84y^3 + 112y^4)t^5 \\ + 128(y + 28y^2 + 224y^3 + 672y^4 + 672y^5)t^6 \\ + 256(y + 40y^2 + 480y^3 + 2400y^4 + 5280y^5 + 4224y^6)t^7 + \cdots . \end{aligned}$$

2.3.3. *Others consequences.* Since Bubble is, by Theorem 2.8, a 2-bubble decomposition of BNC, we can use the results of Section 1.3 to obtain the generating set, the group of symmetries, and the presentation by generators and relations of BNC.

Thus, by Propositions 1.5 and 2.3, the generating set of BNC is the set of the eight BNCs of arity 2.

By Propositions 1.6 and 2.4, the group of symmetries of BNC is generated by the maps $\mathrm{cpl}' := \mathbf{Hull}(\mathrm{cpl})$ and $\mathrm{ret}' := \mathbf{Hull}(\mathrm{ret})$. For any BNC $\mathfrak{C}$, $\mathrm{cpl}'(\mathfrak{C})$ is the BNC obtained by swapping the colours of the red and blue diagonals of $\mathfrak{C}$, and by swapping the colours of the edges of $\mathfrak{C}$. Moreover, for any BNC $\mathfrak{C}$, $\mathrm{ret}'(\mathfrak{C})$ is the BNC obtained by applying on $\mathfrak{C}$ the reflection through the vertical line passing by its base.

Finally, by Proposition 1.7 and Theorem 2.5, BNC admits the presentation by generators and relations of the statement of Theorem 2.5.

3. Suboperads of the operad of bicoloured noncrossing configurations

We now study some of the suboperads of BNC generated by various sets of BNCs. We shall mainly focus on the suboperads generated by sets of two BNCs of arity 2.

3.1. **Overview of the obtained suboperads.** In what follows, we denote by $\langle G\rangle$ the suboperad of BNC generated by a set G of BNCs and, when G is a set of bubbles, by $\langle\langle G\rangle\rangle$ the coloured suboperad of Bubble generated by G.

3.1.1. *Orbits of suboperads.* There are $2^8 = 256$ suboperads of BNC generated by elements of arity 2. The symmetries provided by the group of symmetries of BNC allow to gather some of these. Indeed, if G_1 and G_2 are two sets of BNCs and ϕ is a map of the group of symmetries of BNC such that $\phi(G_1) = G_2$, the suboperads $\langle G_1\rangle$ and $\langle G_2\rangle$ would be isomorphic or antiisomorphic. We say in this case that these two operads are *equivalent*. There are in this way only 88 orbits of suboperads that are pairwise nonequivalent.

3.1.2. *Suboperads on one generator.* There are three orbits of suboperads of BNC generated by one generator of arity 2. The first contains $\langle\triangle\rangle$. By induction on the arity, one can show that this operad contains all the triangulations and that it is free. The second one contains $\langle\triangle\rangle$. By using similar arguments, one can show that this operad is also free and isomorphic to the latter. The third orbit contains $\langle\triangle\rangle$. This operad contains exactly one element of any arity, and hence, is the associative operad.

3.1.3. *Operads of noncrossing trees and plants.* The first named author defined in [Cha07] an operad involving noncrossing trees and an operad involving noncrossing plants. These operads are, directly from the definition, respectively the suboperads $\langle\triangle,\triangle\rangle$ and $\langle\triangle,\triangle,\triangle\rangle$ of BNC. The operad of noncrossing trees governs L-algebras, a sort of algebras introduced by Leroux [Ler11].

3.1.4. *Suboperads on two generators.* The $\binom{8}{2} = 28$ suboperads of BNC generated by two BNCs of arity 2 form eleven orbits. Table 1 summarizes some information about these. Some of these

Operad	Dimensions	Presentation
$\langle\triangle,\triangle\rangle$	1, 2, 8, 40, 224, 1344, 8448, 54912	free
$\langle\triangle,\triangle\rangle$	1, 2, 8, 40, 216, 1246, 7516, 46838	quartic or more
$\langle\triangle,\triangle\rangle$ $\langle\triangle,\triangle\rangle$ $\langle\triangle,\triangle\rangle$	1, 2, 8, 38, 200, 1124, 6608, 40142	cubic
$\langle\triangle,\triangle\rangle$	1, 2, 7, 31, 154, 820, 4575, 26398	quadratic
$\langle\triangle,\triangle\rangle$	1, 2, 7, 30, 143, 728, 3876, 21318	quadratic
$\langle\triangle,\triangle\rangle$ $\langle\triangle,\triangle\rangle$ $\langle\triangle,\triangle\rangle$ $\langle\triangle,\triangle\rangle$	1, 2, 6, 22, 90, 394, 1806, 8558	quadratic

TABLE 1. The eleven orbits of suboperads of BNC generated by two generators of arity 2, their dimensions and the degrees of nontrivial relations between their generators.

operads are well-known operads: the free operad $\langle\triangle,\triangle\rangle$ on two generators of arity 2, the operad of noncrossing trees [Cha07, Ler11] $\langle\triangle,\triangle\rangle$, the dipterous operad [LR03, Zin12] $\langle\triangle,\triangle\rangle$, and the 2-associative operad [LR06, Zin12] $\langle\triangle,\triangle\rangle$. All the Hilbert series of the eleven operads are algebraic, with the genus of the associated algebraic curve being 0. For some of these series, the coefficients form known sequences of [Slo]. The first one of Table 1 is Sequence **A052701**, the fourth is Sequence **A007863**, the fifth is Sequence **A006013**, and the sixth is Sequence **A006318**.

3.1.5. *Suboperads on more than two generators.* Some suboperads of BNC generated by more than two generators are very complicated to study. For instance, the operad $\langle \triangle, \triangle, \triangle \rangle$ has two equivalence classes of nontrivial relations in degree 2, three in degree 3, ten in degree 4 and seems to have no nontrivial relations in higher degree (this has been checked until degree 6). The operad $\langle \triangle, \triangle, \triangle, \triangle \rangle$ is also complicated since it has four equivalence classes of nontrivial relations in degree 2, sixteen in degree 3 and seems to have no nontrivial relations in higher degree (this has been checked until degree 6).

3.2. **Suboperads generated by two elements of arity 2.** For any of the eleven nonequivalent suboperads of BNC generated by two elements of arity 2, we compute its dimensions and provide a presentation by generators and relations by passing through a bubble decomposition of it.

3.2.1. *Outline of the study.* Let $\langle G \rangle$ be one of these operads. Since, by Theorem 2.8, Bubble is a bubble decomposition of BNC and $\langle G \rangle$ is generated by bubbles, $\langle\langle G \rangle\rangle$ is a bubble decomposition of $\langle G \rangle$. We shall compute the dimensions and establish the presentation by generators and relations of $\langle\langle G \rangle\rangle$ to obtain in return, by Propositions 1.3 and 1.7, the dimensions and the presentation by generators and relations of $\langle G \rangle$.

To compute the dimensions of $\langle\langle G \rangle\rangle$, we shall furnish a description of its elements and then deduce from the description its coloured Hilbert series. In what follows, we will only detail this for the first orbit. The computations for the other orbits are analogous. Table 2 shows the first coefficients of the coloured Hilbert series of the eleven coloured suboperads. All of these series are rational.

Coloured operad	Based bubbles	Nonbased bubbles
$\langle\langle \triangle, \triangle \rangle\rangle$	2, 2, 2, 2, 2, 2, 2	0, 0, 0, 0, 0, 0, 0
$\langle\langle \triangle, \triangle \rangle\rangle$	1, 2, 5, 10, 21, 42, 85	1, 2, 5, 10, 21, 42, 85
$\langle\langle \triangle, \triangle \rangle\rangle$ $\langle\langle \triangle, \triangle \rangle\rangle$ $\langle\langle \triangle, \triangle \rangle\rangle$	1, 2, 4, 8, 16, 32, 64	1, 2, 4, 8, 16, 32, 64
$\langle\langle \triangle, \triangle \rangle\rangle$	2, 3, 5, 8, 13, 21, 34	0, 0, 0, 0, 0, 0, 0
$\langle\langle \triangle, \triangle \rangle\rangle$	2, 3, 4, 5, 6, 7, 8	0, 0, 0, 0, 0, 0, 0
$\langle\langle \triangle, \triangle \rangle\rangle$	2, 4, 8, 16, 32, 64, 128	0, 0, 0, 0, 0, 0, 0
$\langle\langle \triangle, \triangle \rangle\rangle$ $\langle\langle \triangle, \triangle \rangle\rangle$ $\langle\langle \triangle, \triangle \rangle\rangle$	1, 1, 1, 1, 1, 1, 1	1, 1, 1, 1, 1, 1, 1

TABLE 2. The eleven orbits of 2-coloured suboperads of Bubble generated by two generators of arity 2 and the number of their bubbles, based and nonbased.

To establish the presentation of $\langle\langle G \rangle\rangle$, we shall use the same strategy as the one used for the proof of the presentation of Bubble (see the proof of Theorem 2.5). Recall that this consists in exhibiting an orientation $\mapsto$ of the presentation we want to prove such that $\mapsto$ is a terminating rewrite rule on coloured syntax trees and its normal forms are in bijection with the elements of $\langle\langle G \rangle\rangle$. We call these rewrite rules *good orientations*. In what follows, we will exhibit a good orientation for any of the studied operads, except the first and the third ones.

3.2.2. *First orbit.* This orbit consists in the operads $\langle \triangle, \triangle \rangle$, $\langle \triangle, \triangle \rangle$, $\langle \triangle, \triangle \rangle$, and $\langle \triangle, \triangle \rangle$. We choose $\langle \triangle, \triangle \rangle$ as representative of the orbit.

Proposition 3.1. *The set of bubbles of $\langle\langle \triangle, \triangle \rangle\rangle$ is the set of based bubbles whose all edges of the border except possibly the last are blue. Moreover, the coloured Hilbert series of $\langle\langle \triangle, \triangle \rangle\rangle$ satisfy*

$$\mathrm{B}_1(z_1, z_2) = \frac{z_1 z_2 + z_2^2}{1 - z_2} \qquad \textit{and} \qquad \mathrm{B}_2(z_1, z_2) = 0. \tag{3.2.1}$$

Proof. Let us prove by induction on the arity that any bubble of $\langle\langle \triangle, \triangle \rangle\rangle$ satisfies the statement of the Proposition. The two bubbles $\triangle$ and $\triangle$ of arity 2 satisfy the statement. Let $\mathfrak{B}$ be a bubble of arity $n \geqslant 3$ satisfying the statement, $i \in [n]$, and $\mathfrak{B}' := \mathfrak{B} \circ_i g$ where $g \in \{\triangle, \triangle\}$. Then, since the composition $\mathfrak{B} \circ_i g$ is well-defined, we have $i = n$ and the last edge of $\mathfrak{B}$ is uncoloured. Thus, $\mathfrak{B}'$ is obtained from $\mathfrak{B}$ by replacing its last uncoloured edge by two blue edges of by a blue edge followed by an uncoloured edge, whether the colour of the last edge of g. This shows that any bubble of $\langle\langle \triangle, \triangle \rangle\rangle$ satisfies the statement.

Conversely, let us prove by induction on the arity that any bubble satisfying the statement of the Proposition belongs to $\langle\langle \triangle, \triangle \rangle\rangle$. This is true for the bubbles $\triangle$ and $\triangle$ since they are generators of $\langle\langle \triangle, \triangle \rangle\rangle$. Let $\mathfrak{B}$ be a bubble of arity $n \geqslant 3$ satisfying the statement and $\mathfrak{B}'$ be the bubble of arity $n-1$ obtained by replacing the two last edges of $\mathfrak{B}$ by an uncoloured edge. If the last edge of $\mathfrak{B}$ is blue, we have $\mathfrak{B} = \mathfrak{B}' \circ_{n-1} \triangle$ and otherwise, $\mathfrak{B} = \mathfrak{B}' \circ_{n-1} \triangle$. Since $\mathfrak{B}'$ satisfies the statement, it is by induction hypothesis, an element of $\langle\langle \triangle, \triangle \rangle\rangle$. Thus, $\mathfrak{B}$ also is.

Finally, the expressions for the coloured Hilbert series of $\langle\langle \triangle, \triangle \rangle\rangle$ follow directly from the above description of its elements. $\square$

Proposition 3.2. *The Hilbert series* F *of* $\langle \triangle, \triangle \rangle$ *satisfies*

$$t - \mathrm{F} + 2\,\mathrm{F}^2 = 0. \tag{3.2.2}$$

Theorem 3.3. *The operad $\langle \triangle, \triangle \rangle$ is the free operad generated by two generators of arity 2.*

Proof. The elements of arity n of the free operad $\mathcal{P}$ generated by two generators of arity 2 are binary trees with n leaves and such that internal nodes can be labeled in two different ways. Hence, the Hilbert series F of $\mathcal{P}$ satisfies (3.2.2) and by Proposition 3.2, $\langle \triangle, \triangle \rangle$ and $\mathcal{P}$ have the same Hilbert series. Thus, since there is no nontrivial relation relation between the generators of $\langle \triangle, \triangle \rangle$ in degree 2, there is no nontrivial relation in $\langle \triangle, \triangle \rangle$ of higher degree. Then, $\langle \triangle, \triangle \rangle$ and $\mathcal{P}$ are isomorphic. $\square$

3.2.3. *Second orbit.* This orbit consists in the operad $\langle \triangle, \triangle \rangle$.

Proposition 3.4. *The set of based (resp. nonbased) bubbles of $\langle\langle \triangle, \triangle \rangle\rangle$ of arity n is the set of based (resp. nonbased) bubbles having at least two consecutive edges of the border of a same colour and the number of blue (resp. uncoloured) edges of the border is congruent to $1-n$ modulo 3. Moreover, the coloured Hilbert series of $\langle\langle \triangle, \triangle \rangle\rangle$ satisfy*

$$\mathrm{B}_1(z_1, z_2) = \frac{z_1 + z_2^2}{1 - 3z_1 z_2 - z_1^3 - z_2^3} - \frac{z_2}{1 - z_1 z_2} \tag{3.2.3}$$

and

$$\mathrm{B}_2(z_1, z_2) = \frac{z_2 + z_1^2}{1 - 3z_1 z_2 - z_1^3 - z_2^3} - \frac{z_1}{1 - z_1 z_2}. \tag{3.2.4}$$

Proposition 3.5. *The Hilbert series* F *of* $\langle \color{blue}\triangle, \color{gray}\triangle \rangle$ *satisfies*

$$4t - 2t^2 - t^3 + t^4 + (-4 + 4t - t^2 + 2t^3)\,\mathrm{F} + (6 + t)\,\mathrm{F}^2 + (1 - 2t)\,\mathrm{F}^3 - \mathrm{F}^4 = 0. \tag{3.2.5}$$

Proposition 3.6. *The operad* $\langle \color{blue}\triangle, \color{gray}\triangle \rangle$ *does not admit nontrivial relations between its generators in degree two, three, five and six. It admits the following non trivial relations between its generators in degree four:*

$$((\color{blue}\triangle \color{black}\circ_2 \color{gray}\triangle\color{black}) \circ_3 \color{blue}\triangle\color{black}) \circ_3 \color{gray}\triangle \color{black} = ((\color{blue}\triangle \color{black}\circ_1 \color{gray}\triangle\color{black}) \circ_1 \color{blue}\triangle\color{black}) \circ_2 \color{gray}\triangle\color{black}, \tag{3.2.6}$$

$$((\color{blue}\triangle \color{black}\circ_2 \color{gray}\triangle\color{black}) \circ_2 \color{blue}\triangle\color{black}) \circ_4 \color{blue}\triangle \color{black} = ((\color{blue}\triangle \color{black}\circ_1 \color{gray}\triangle\color{black}) \circ_1 \color{blue}\triangle\color{black}) \circ_3 \color{blue}\triangle\color{black}, \tag{3.2.7}$$

$$((\color{blue}\triangle \color{black}\circ_2 \color{gray}\triangle\color{black}) \circ_2 \color{blue}\triangle\color{black}) \circ_3 \color{gray}\triangle \color{black} = ((\color{blue}\triangle \color{black}\circ_1 \color{gray}\triangle\color{black}) \circ_1 \color{blue}\triangle\color{black}) \circ_4 \color{gray}\triangle\color{black}, \tag{3.2.8}$$

$$((\color{blue}\triangle \color{black}\circ_1 \color{gray}\triangle\color{black}) \circ_3 \color{gray}\triangle\color{black}) \circ_4 \color{blue}\triangle \color{black} = ((\color{blue}\triangle \color{black}\circ_1 \color{gray}\triangle\color{black}) \circ_2 \color{blue}\triangle\color{black}) \circ_2 \color{gray}\triangle\color{black}, \tag{3.2.9}$$

$$((\color{gray}\triangle \color{black}\circ_2 \color{blue}\triangle\color{black}) \circ_3 \color{gray}\triangle\color{black}) \circ_3 \color{blue}\triangle \color{black} = ((\color{gray}\triangle \color{black}\circ_1 \color{blue}\triangle\color{black}) \circ_1 \color{gray}\triangle\color{black}) \circ_2 \color{blue}\triangle\color{black}, \tag{3.2.10}$$

$$((\color{gray}\triangle \color{black}\circ_2 \color{blue}\triangle\color{black}) \circ_2 \color{gray}\triangle\color{black}) \circ_4 \color{gray}\triangle \color{black} = ((\color{gray}\triangle \color{black}\circ_1 \color{blue}\triangle\color{black}) \circ_1 \color{gray}\triangle\color{black}) \circ_3 \color{gray}\triangle\color{black}, \tag{3.2.11}$$

$$((\color{gray}\triangle \color{black}\circ_2 \color{blue}\triangle\color{black}) \circ_2 \color{gray}\triangle\color{black}) \circ_3 \color{blue}\triangle \color{black} = ((\color{gray}\triangle \color{black}\circ_1 \color{blue}\triangle\color{black}) \circ_1 \color{gray}\triangle\color{black}) \circ_4 \color{blue}\triangle\color{black}, \tag{3.2.12}$$

$$((\color{gray}\triangle \color{black}\circ_1 \color{blue}\triangle\color{black}) \circ_3 \color{blue}\triangle\color{black}) \circ_4 \color{gray}\triangle \color{black} = ((\color{gray}\triangle \color{black}\circ_1 \color{blue}\triangle\color{black}) \circ_2 \color{gray}\triangle\color{black}) \circ_2 \color{blue}\triangle\color{black}. \tag{3.2.13}$$

Proof. This statement is proven with the help of the computer. All compositions between the generators $\color{blue}\triangle$ and $\color{gray}\triangle$ are computed up to degree six and relations thus established. □

Proposition 3.6 does not provide a presentation by generators and relations of $\langle \color{blue}\triangle, \color{gray}\triangle \rangle$. The methods employed in this article fail to establish the presentation of $\langle\langle \color{blue}\triangle, \color{gray}\triangle \rangle\rangle$ because it is not possible to define a good orientation of the relations of the statement of Proposition 3.6. Indeed, in degree six, all the orientations have no less than 7518 normal forms whereas they should be 7516. Nevertheless, these relations seem to be the only nontrivial ones; this may be proved by using the Knuth-Bendix completion algorithm (see [KB70,BN98]) over an appropriate orientation of the relations.

3.2.4. *Third orbit.* This orbit consists in the operads $\langle \color{blue}\triangle, \triangle \rangle$, $\langle \color{blue}\triangle, \triangle \rangle$, $\langle \triangle, \color{gray}\triangle \rangle$, and $\langle \triangle, \color{gray}\triangle \rangle$. We choose $\langle \color{blue}\triangle, \triangle \rangle$ as representative of the orbit.

Proposition 3.7. *The set of based (resp. nonbased) bubbles of* $\langle\langle \color{blue}\triangle, \triangle \rangle\rangle$ *of arity* n *is the set of based (resp. nonbased) bubbles whose first edge is blue and the number of uncoloured edges of the border is congruent to* n *(resp.* $n+1$*) modulo 2. Moreover, the coloured Hilbert series of* $\langle\langle \color{blue}\triangle, \triangle \rangle\rangle$ *satisfy*

$$\mathrm{B}_1(z_1, z_2) = \frac{z_2^2}{1 - 2z_1 + z_1^2 - z_2^2} \qquad \textit{and} \qquad \mathrm{B}_2(z_1, z_2) = \frac{z_1 z_2 - z_1^2 z_2 + z_2^3}{1 - 2z_1 + z_1^2 - z_2^2}. \tag{3.2.14}$$

Proposition 3.8. *The Hilbert series* F *of* $\langle \color{blue}\triangle, \triangle \rangle$ *satisfies*

$$2t - t^2 + (2t - 2)\,\mathrm{F} + 3\,\mathrm{F}^2 = 0. \tag{3.2.15}$$

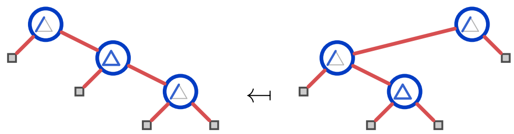
Theorem 3.9. *The operad* $\langle \blacktriangle, \triangle \rangle$ *admits the presentation* $(\{\blacktriangle, \triangle\}, \leftrightarrow)$ *where* $\leftrightarrow$ *is the equivalence relation satisfying*

$$(\mathrm{c}(\triangle) \circ_2 \mathrm{c}(\blacktriangle)) \circ_3 \mathrm{c}(\triangle) \quad \leftrightarrow \quad (\mathrm{c}(\triangle) \circ_1 \mathrm{c}(\triangle)) \circ_2 \mathrm{c}(\blacktriangle), \tag{3.2.16}$$

$$(\mathrm{c}(\blacktriangle) \circ_2 \mathrm{c}(\triangle)) \circ_3 \mathrm{c}(\blacktriangle) \quad \leftrightarrow \quad (\mathrm{c}(\blacktriangle) \circ_1 \mathrm{c}(\triangle)) \circ_2 \mathrm{c}(\blacktriangle). \tag{3.2.17}$$

The proof of Theorem 3.9 relies on the good orientation

(3.2.18)



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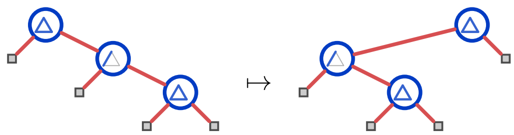
(3.2.19)



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From the above presentation, we deduce that any $\langle \blacktriangle, \triangle \rangle$-algebra is a set S equipped with two binary operations $\diamond$ and $\bullet$ satisfying, for any $x, y, z, t \in S$,

$$x \diamond (y \bullet (z \diamond t)) = (x \diamond (y \bullet z)) \diamond t, \tag{3.2.20}$$

$$x \bullet (y \diamond (z \bullet t)) = (x \diamond (y \bullet z)) \bullet t. \tag{3.2.21}$$

3.2.5. *Fourth orbit.* This orbit consists in the operads $\langle \triangle, \triangle \rangle$ and $\langle \triangle, \triangle \rangle$. We choose $\langle \triangle, \triangle \rangle$ as representative of the orbit.

Proposition 3.10. *The set of bubbles of* $\langle\langle \triangle, \triangle \rangle\rangle$ *is the set of bubbles whose first edge is blue and last edge uncoloured. Moreover, the coloured Hilbert series of* $\langle\langle \triangle, \triangle \rangle\rangle$ *satisfy*

$$\mathrm{B}_1(z_1, z_2) = \frac{z_1 z_2}{1 - z_1 - z_2} \qquad \textit{and} \qquad \mathrm{B}_2(z_1, z_2) = \frac{z_1 z_2}{1 - z_1 - z_2}. \tag{3.2.22}$$

Proposition 3.11. *The Hilbert series* F *of* $\langle \triangle, \triangle \rangle$ *satisfies*

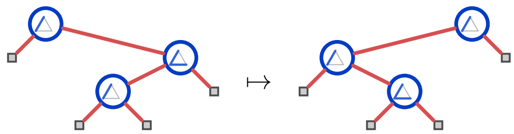
$$2t - t^2 + (2t - 2)\,\mathrm{F} + 3\,\mathrm{F}^2 = 0. \tag{3.2.23}$$

Theorem 3.12. *The operad* $\langle \triangle, \triangle \rangle$ *admits the presentation* $(\{\triangle, \triangle\}, \leftrightarrow)$ *where* $\leftrightarrow$ *is the equivalence relation satisfying*

$$(\mathrm{c}(\triangle) \circ_2 \mathrm{c}(\triangle)) \circ_2 \mathrm{c}(\triangle) \quad \leftrightarrow \quad (\mathrm{c}(\triangle) \circ_1 \mathrm{c}(\triangle)) \circ_2 \mathrm{c}(\triangle), \tag{3.2.24}$$

$$(\mathrm{c}(\triangle) \circ_2 \mathrm{c}(\triangle)) \circ_2 \mathrm{c}(\triangle) \quad \leftrightarrow \quad (\mathrm{c}(\triangle) \circ_1 \mathrm{c}(\triangle)) \circ_2 \mathrm{c}(\triangle). \tag{3.2.25}$$

The proof of Theorem 3.12 relies on the good orientation

(3.2.26)

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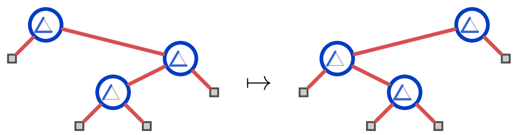
(3.2.27)

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From the above presentation, we deduce that any $\langle \triangle, \triangle \rangle$-algebra is a set S equipped with two binary operations $\diamond$ and $\bullet$ satisfying, for any $x, y, z, t \in S$,

$$x \diamond ((y \diamond z) \bullet t) = (x \diamond (y \bullet z)) \diamond t, \tag{3.2.28}$$

$$x \bullet ((y \diamond z) \bullet t) = (x \diamond (y \bullet z)) \bullet t. \tag{3.2.29}$$

3.2.6. *Fifth orbit.* This orbit consists in the operads $\langle \triangle, \triangle \rangle$ and $\langle \triangle, \triangle \rangle$. We choose $\langle \triangle, \triangle \rangle$ as representative of the orbit.

Proposition 3.13. *The set of based (resp. nonbased) bubbles of $\langle\langle \triangle, \triangle \rangle\rangle$ is the set of based (resp. nonbased) bubbles whose penultimate edge is blue (resp. uncoloured) and the last edge is uncoloured (resp. blue). Moreover, the coloured Hilbert series of $\langle\langle \triangle, \triangle \rangle\rangle$ satisfy*

$$\mathrm{B}_1(z_1, z_2) = \frac{z_1 z_2}{1 - z_1 - z_2} \qquad \textit{and} \qquad \mathrm{B}_2(z_1, z_2) = \frac{z_1 z_2}{1 - z_1 - z_2}. \tag{3.2.30}$$

Proposition 3.14. *The Hilbert series* F *of* $\langle \triangle, \triangle \rangle$ *satisfies*

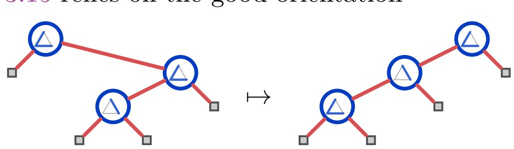
$$2t - t^2 + (2t - 2)\,\mathrm{F} + 3\,\mathrm{F}^2 = 0. \tag{3.2.31}$$

Theorem 3.15. *The operad $\langle \triangle, \triangle \rangle$ admits the presentation $(\{\triangle, \triangle\}, \leftrightarrow)$ where $\leftrightarrow$ is the equivalence relation satisfying*

$$(\mathrm{c}(\triangle) \circ_2 \mathrm{c}(\triangle)) \circ_2 \mathrm{c}(\triangle) \quad \leftrightarrow \quad (\mathrm{c}(\triangle) \circ_1 \mathrm{c}(\triangle)) \circ_1 \mathrm{c}(\triangle), \tag{3.2.32}$$

$$(\mathrm{c}(\triangle) \circ_2 \mathrm{c}(\triangle)) \circ_2 \mathrm{c}(\triangle) \quad \leftrightarrow \quad (\mathrm{c}(\triangle) \circ_1 \mathrm{c}(\triangle)) \circ_1 \mathrm{c}(\triangle). \tag{3.2.33}$$

The proof of Theorem 3.15 relies on the good orientation

(3.2.34)

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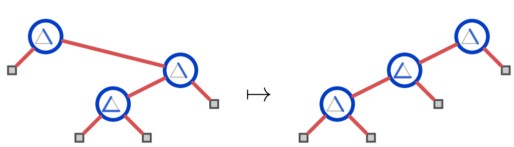
(3.2.35)

.

From the above presentation, we deduce that any $\langle \triangle, \triangle \rangle$-algebra is a set S equipped with two binary operations $\diamond$ and $\bullet$ satisfying, for any $x, y, z, t \in S$,

$$x \bullet ((y \diamond z) \bullet t) = ((x \bullet y) \diamond z) \bullet t, \tag{3.2.36}$$

$$x \diamond ((y \bullet z) \diamond t) = ((x \diamond y) \bullet z) \diamond t. \tag{3.2.37}$$

3.2.7. *Sixth orbit.* This orbit consists in the operads $\langle \triangle, \triangle \rangle$ and $\langle \triangle, \triangle \rangle$. We choose $\langle \triangle, \triangle \rangle$ as representative of the orbit.

Proposition 3.16. *The set of bubbles of $\langle\langle \triangle, \triangle \rangle\rangle$ is the set of based bubbles whose maximal sequences of blues edges of the border have even length. Moreover, the coloured Hilbert series of $\langle\langle \triangle, \triangle \rangle\rangle$ satisfy*

$$\mathrm{B}_1(z_1, z_2) = \frac{z_1^2 + z_2^2 + z_1 z_2^2}{1 - z_1 - z_2^2} \qquad \textit{and} \qquad \mathrm{B}_2(z_1, z_2) = 0. \tag{3.2.38}$$

Proposition 3.17. *The Hilbert series* F *of* $\langle \triangle, \triangle \rangle$ *satisfies*

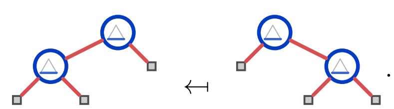
$$t + (t-1)\,\mathrm{F} + \mathrm{F}^2 + \mathrm{F}^3 = 0. \tag{3.2.39}$$

Theorem 3.18. *The operad $\langle \triangle, \triangle \rangle$ admits the presentation $(\{\triangle, \triangle\}, \leftrightarrow)$ where $\leftrightarrow$ is the equivalence relation satisfying*

$$\mathrm{c}(\triangle) \circ_1 \mathrm{c}(\triangle) \quad \leftrightarrow \quad \mathrm{c}(\triangle) \circ_2 \mathrm{c}(\triangle). \tag{3.2.40}$$

The proof of Theorem 3.18 relies on the good orientation

(3.2.41)



From the above presentation, we deduce that any $\langle \triangle, \triangle \rangle$-algebra is a set S equipped with two binary operations $\diamond$ and $\bullet$ satisfying, for any $x, y, z \in S$,

$$(x \diamond y) \diamond z = x \diamond (y \diamond z). \tag{3.2.42}$$

3.2.8. *Seventh orbit.* This orbit consists in the operads $\langle \triangle, \triangle \rangle$ and $\langle \triangle, \triangle \rangle$. We choose $\langle \triangle, \triangle \rangle$ as representative of the orbit.

Proposition 3.19. *The set of bubbles of $\langle\langle \triangle, \triangle \rangle\rangle$ is the set of based bubbles having exactly one uncoloured edge in the border. Moreover, the coloured Hilbert series of $\langle\langle \triangle, \triangle \rangle\rangle$ satisfy*

$$\mathrm{B}_1(z_1, z_2) = \frac{2 z_1 z_2 - z_1 z_2^2}{(1 - z_2)^2} \qquad \textit{and} \qquad \mathrm{B}_2(z_1, z_2) = 0. \tag{3.2.43}$$

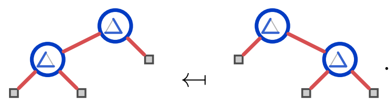
Proposition 3.20. *The Hilbert series* F *of* $\langle \triangle, \triangle \rangle$ *satisfies*

$$t - \mathrm{F} + 2\,\mathrm{F}^2 - \mathrm{F}^3 = 0. \tag{3.2.44}$$

Theorem 3.21. *The operad* $\langle \boldsymbol{\triangle}, \boldsymbol{\triangle} \rangle$ *admits the presentation* $(\{\boldsymbol{\triangle}, \boldsymbol{\triangle}\}, \leftrightarrow)$ *where* $\leftrightarrow$ *is the equivalence relation satisfying*

$$\mathrm{c}(\boldsymbol{\triangle}) \circ_1 \mathrm{c}(\boldsymbol{\triangle}) \quad \leftrightarrow \quad \mathrm{c}(\boldsymbol{\triangle}) \circ_2 \mathrm{c}(\boldsymbol{\triangle}). \tag{3.2.45}$$

The proof of Theorem 3.21 relies on the good orientation

(3.2.46) 

From the above presentation, we deduce that any $\langle \boldsymbol{\triangle}, \boldsymbol{\triangle} \rangle$-algebra is a set S equipped with two binary operations $\diamond$ and $\bullet$ satisfying, for any $x, y, z \in S$,

$$(x \bullet y) \diamond z = x \bullet (y \diamond z). \tag{3.2.47}$$

This relation is the one of L-algebras [Cha07, Ler11].

3.2.9. *Eighth orbit.* This orbit consists in the operads $\langle \boldsymbol{\triangle}, \triangle \rangle$, $\langle \boldsymbol{\triangle}, \triangle \rangle$, $\langle \boldsymbol{\triangle}, \triangle \rangle$, and $\langle \boldsymbol{\triangle}, \triangle \rangle$. We choose $\langle \boldsymbol{\triangle}, \triangle \rangle$ as representative of the orbit.

Proposition 3.22. *The set of bubbles of* $\langle\langle \boldsymbol{\triangle}, \triangle \rangle\rangle$ *is the set of based bubbles whose last edge is uncoloured. Moreover, the coloured Hilbert series of* $\langle\langle \boldsymbol{\triangle}, \triangle \rangle\rangle$ *satisfy*

$$\mathrm{B}_1(z_1, z_2) = \frac{z_1^2 + z_1 z_2}{1 - z_1 - z_2} \qquad \textit{and} \qquad \mathrm{B}_2(z_1, z_2) = 0. \tag{3.2.48}$$

Proposition 3.23. *The Hilbert series* F *of* $\langle \boldsymbol{\triangle}, \triangle \rangle$ *satisfies*

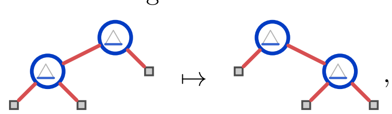
$$t - (1 - t)\,\mathrm{F} + \mathrm{F}^2 = 0. \tag{3.2.49}$$

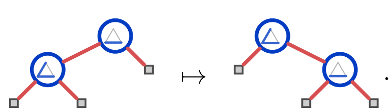
Theorem 3.24. *The operad* $\langle \boldsymbol{\triangle}, \triangle \rangle$ *admits the presentation* $(\{\boldsymbol{\triangle}, \triangle\}, \leftrightarrow)$ *where* $\leftrightarrow$ *is the equivalence relation satisfying*

$$\mathrm{c}(\triangle) \circ_1 \mathrm{c}(\triangle) \quad \leftrightarrow \quad \mathrm{c}(\triangle) \circ_2 \mathrm{c}(\triangle), \tag{3.2.50}$$

$$\mathrm{c}(\triangle) \circ_1 \mathrm{c}(\boldsymbol{\triangle}) \quad \leftrightarrow \quad \mathrm{c}(\boldsymbol{\triangle}) \circ_2 \mathrm{c}(\triangle). \tag{3.2.51}$$

The proof of Theorem 3.24 relies on the good orientation

(3.2.52) 

(3.2.53) 

From the above presentation, we deduce that any $\langle \boldsymbol{\triangle}, \triangle \rangle$-algebra is a set S equipped with two binary operations $\diamond$ and $\bullet$ satisfying, for any $x, y, z \in S$,

$$(x \diamond y) \diamond z = x \diamond (y \diamond z), \tag{3.2.54}$$

$$(x \bullet y) \diamond z = x \bullet (y \diamond z). \tag{3.2.55}$$

These relations are similar to the ones of duplicial algebras [Lod08,Zin12]. In a duplicial algebra, $\bullet$ is associative in addition.

3.2.10. *Ninth orbit.* This orbit consists in the operads $\langle \triangle, \triangle \rangle$ and $\langle \triangle, \triangle \rangle$. We choose $\langle \triangle, \triangle \rangle$ as representative of the orbit.

Proposition 3.25. *The set of bubbles of $\langle\langle \triangle, \triangle \rangle\rangle$ is the set of bubbles whose all edges of the border are blue. Moreover, the coloured Hilbert series of $\langle\langle \triangle, \triangle \rangle\rangle$ satisfy*

$$\mathrm{B}_1(z_1, z_2) = \frac{z_2^2}{1 - z_2} \qquad \textit{and} \qquad \mathrm{B}_2(z_1, z_2) = \frac{z_2^2}{1 - z_2}. \tag{3.2.56}$$

Proposition 3.26. *The Hilbert series F of $\langle \triangle, \triangle \rangle$ satisfies*

$$t - (1 - t)\,\mathrm{F} + \mathrm{F}^2 = 0. \tag{3.2.57}$$

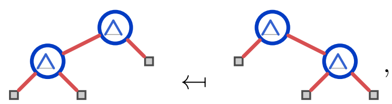
Theorem 3.27. *The operad $\langle \triangle, \triangle \rangle$ admits the presentation $(\{\triangle, \triangle\}, \leftrightarrow)$ where $\leftrightarrow$ is the equivalence relation satisfying*

$$\mathrm{c}(\triangle) \circ_1 \mathrm{c}(\triangle) \quad \leftrightarrow \quad \mathrm{c}(\triangle) \circ_2 \mathrm{c}(\triangle), \tag{3.2.58}$$

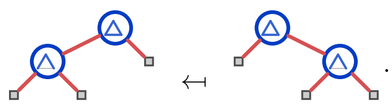
$$\mathrm{c}(\triangle) \circ_1 \mathrm{c}(\triangle) \quad \leftrightarrow \quad \mathrm{c}(\triangle) \circ_2 \mathrm{c}(\triangle). \tag{3.2.59}$$

The proof of Theorem 3.27 relies on the good orientation

(3.2.60)



(3.2.61)



From the above presentation, we deduce that any $\langle \triangle, \triangle \rangle$-algebra is a set S equipped with two binary operations $\diamond$ and $\bullet$ satisfying, for any $x, y, z \in S$,

$$(x \diamond y) \diamond z = x \diamond (y \diamond z), \tag{3.2.62}$$

$$(x \diamond y) \bullet z = x \bullet (y \diamond z). \tag{3.2.63}$$

3.2.11. *Tenth orbit.* This orbit consists in the operads $\langle \triangle, \triangle \rangle$, $\langle \triangle, \triangle \rangle$, $\langle \triangle, \triangle \rangle$, and $\langle \triangle, \triangle \rangle$. We choose $\langle \triangle, \triangle \rangle$ as representative of the orbit.

Proposition 3.28. *The set of based (resp. nonbased) bubbles of $\langle\langle \triangle, \triangle \rangle\rangle$ is the set of based (resp. nonbased) bubbles whose first edge is uncoloured (resp. blue) and the other edges of the border are blue. Moreover, the coloured Hilbert series of $\langle\langle \triangle, \triangle \rangle\rangle$ satisfy*

$$\mathrm{B}_1(z_1, z_2) = \frac{z_1 z_2}{1 - z_2} \qquad \textit{and} \qquad \mathrm{B}_2(z_1, z_2) = \frac{z_2^2}{1 - z_2}. \tag{3.2.64}$$

Proposition 3.29. *The Hilbert series F of $\langle \triangle, \triangle \rangle$ satisfies*

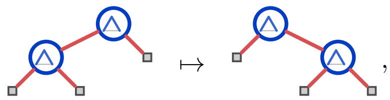
$$t - (1 - t)\,\mathrm{F} + \mathrm{F}^2 = 0. \tag{3.2.65}$$

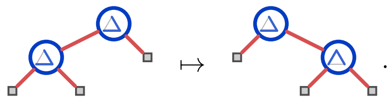
Theorem 3.30. *The operad* $\langle \triangle, \boldsymbol{\triangle} \rangle$ *admits the presentation* $(\{\triangle, \boldsymbol{\triangle}\}, \leftrightarrow)$ *where* $\leftrightarrow$ *is the equivalence relation satisfying*

$$\mathrm{c}(\boldsymbol{\triangle}) \circ_1 \mathrm{c}(\boldsymbol{\triangle}) \quad \leftrightarrow \quad \mathrm{c}(\boldsymbol{\triangle}) \circ_2 \mathrm{c}(\boldsymbol{\triangle}), \tag{3.2.66}$$

$$\mathrm{c}(\triangle) \circ_1 \mathrm{c}(\triangle) \quad \leftrightarrow \quad \mathrm{c}(\triangle) \circ_2 \mathrm{c}(\boldsymbol{\triangle}). \tag{3.2.67}$$

The proof of Theorem 3.30 relies on the good orientation

(3.2.68)


(3.2.69)


From the above presentation, we deduce that any $\langle \triangle, \boldsymbol{\triangle} \rangle$-algebra is a set S equipped with two binary operations $\diamond$ and $\bullet$ satisfying, for any $x, y, z \in S$,

$$(x \diamond y) \diamond z = x \diamond (y \diamond z), \tag{3.2.70}$$

$$(x \bullet y) \bullet z = x \bullet (y \diamond z). \tag{3.2.71}$$

These relations are the ones of dipterous algebras [LR03, Zin12].

3.2.12. *Eleventh orbit.* This orbit consists in the operad $\langle \underline{\triangle}, \boldsymbol{\triangle} \rangle$.

Proposition 3.31. *The set of based (resp. nonbased) bubbles of* $\langle\langle \underline{\triangle}, \boldsymbol{\triangle} \rangle\rangle$ *is the set of based (resp. nonbased) bubbles whose all edges of the border are uncoloured (resp. blue). Moreover, the coloured Hilbert series of* $\langle\langle \underline{\triangle}, \boldsymbol{\triangle} \rangle\rangle$ *satisfy*

$$\mathrm{B}_1(z_1, z_2) = \frac{z_1^2}{1 - z_1} \qquad \textit{and} \qquad \mathrm{B}_2(z_1, z_2) = \frac{z_2^2}{1 - z_2}. \tag{3.2.72}$$

Proposition 3.32. *The Hilbert series* F *of* $\langle \underline{\triangle}, \boldsymbol{\triangle} \rangle$ *satisfies*

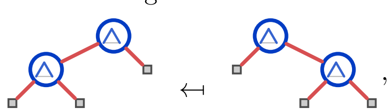
$$t - (1 - t)\,\mathrm{F} + \mathrm{F}^2 = 0. \tag{3.2.73}$$

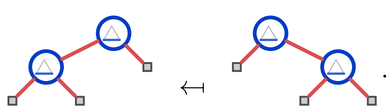
Theorem 3.33. *The operad* $\langle \underline{\triangle}, \boldsymbol{\triangle} \rangle$ *admits the presentation* $(\{\underline{\triangle}, \boldsymbol{\triangle}\}, \leftrightarrow)$ *where* $\leftrightarrow$ *is the equivalence relation satisfying*

$$\mathrm{c}(\boldsymbol{\triangle}) \circ_1 \mathrm{c}(\boldsymbol{\triangle}) \quad \leftrightarrow \quad \mathrm{c}(\boldsymbol{\triangle}) \circ_2 \mathrm{c}(\boldsymbol{\triangle}), \tag{3.2.74}$$

$$\mathrm{c}(\underline{\triangle}) \circ_1 \mathrm{c}(\underline{\triangle}) \quad \leftrightarrow \quad \mathrm{c}(\underline{\triangle}) \circ_2 \mathrm{c}(\underline{\triangle}). \tag{3.2.75}$$

The proof of Theorem 3.33 relies on the good orientation

(3.2.76)


(3.2.77)


From the above presentation, we deduce that any $\langle \triangle, \boldsymbol{\triangle} \rangle$-algebra is a set S equipped with two binary operations $\diamond$ and $\bullet$ satisfying, for any $x, y, z \in S$,

$$(x \diamond y) \diamond z = x \diamond (y \diamond z), \tag{3.2.78}$$

$$(x \bullet y) \bullet z = x \bullet (y \bullet z). \tag{3.2.79}$$

These relations are the ones of two-associative algebras [LR06, Zin12].

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